**Lorentz Transformations**

[looks like using Natural units where set ℏ = c = 1] So we know how scalars, vectors, tensors in general, transform under a Lorentz coordinate transformation. Well actually, they don’t change at all; rather their components change. So rephrasing, we know how the components of vectors, components of tensors in general, transform under a Lorentz transformation. But we’ve now introduced another entity – a four component Dirac spinor, and we’d like to know how its components may change under a Lorentz transformation. Note that even though the spinor has four components, it isn’t a space-time vector (there is no time-like, space-like component of the spinor, and it doesn’t *point* anywhere, it’s just a two-component electron wavefunction tacked on top of a two-component electron-hole wavefunction). So it’s not going to, presumably, transform like a space-time vector does. Nonetheless, we would expect it to transform, as it represents a spin, which has relativistic properties.

Incidentally, instead of treating the Lorentz transformation as a passive coordinate change, we’re going to consider it as an active transformation, whereby we keep our coordinate system unchanged, but boost the spinor into a state moving with speed **v** and rotated by **θ**. This is similar in spirit to how we treated spatial translation and rotation as active transformations whereby we changed the ket itself, not our coordinate system. Let’s call our Lorentz operator which boosts/rotates our spinor (**β**,**θ**). I think we may imagine that it splits into two commuting parts – a spinor (S) part and a spatio/temporal position (P) part:



When we worked out the general Lorentz transformation in the CM folder, the specification we made was that the representation preserve the metric. That will be our starting point here too. Let’s say we represent the general (4D spatial) Lorentz transformation as (Einstein summation implicit):



The Jαβ guys are called the *generators* of the transformation and constitute sixteen 4D matrices in this context. They are an antisymmetric set of matrices so that Jmn = -Jnm (for m≠n) (they have to be because there can be only 6 dof and they cannot cancel out). Note we could write this as:



and we see that there are really only six d.o.f. here. Making contact with our CM Lorentz II file, these matrices would be:



where,



Anyway, the overall requirement for these matrices was preservation of the metric. In terms of our general form, this requirement reduces to, to second order:



Working this out, we find, ultimately, so I hear:



Should be interpreting the η’s as a set of numbers, and J’s as set of matrices. We may verify that a solution is:



You can verify that this does indeed return to us the S, K matrices. Anyway, generalizing to HS by replacing the 4D J matrices with operators, we say that we will have found a representation of the Lorentz transformation in the HS if our J operators similarly satisfy these commutation relations.

**Spacetime transformation**

So what are the J operators in the position HS? It’s probably easiest to extrapolate from our knowledge of the pure rotation case, than to try to work out what operators satisfy these relations. In the pure rotation case we should have:



and so we have:



we can invert this relationship using the identity (see Tensor file)



So,



What we should glean from this will be more clear if we break L into its x, p parts:



(implicit summation as always) Using our identity again, this works out to:



At this point, it might occur to us to generalize to space-time coordinates. So I think a HS representation of the spacetime Lorentz transformation (boost/rotation) is given by:



where (using c = 1 units). And we have the ancillary definitions:



(t isn’t really an operator, but….) We can note that this checks out in two separate cases at least – a pure rotation:



as we expect by design. Acting this on a wavefunction, and projecting against position space we’d have:



as we’ll recall from the Rotation file. And I’m still writing ψ as |ψ> because it’s a spinor and so projecting against space still leaves it a ket in spinor space. And now let’s consider a boost in the purely x-direction for simplicity,



Let’s project its action on a ket onto position space:



To second order in β this is:



where in the penultimate line we treat the x and t behind the derivatives as commuting with ∂x and ∂t, i.e., as constants of a sort. This is of Taylor series form, so the last line follows therewith. Now let’s compare with the familiar Lorentz boost on the spacetime coordinate (t,x):



Presuming this to hold at all orders, and in all directions, we may say:



Perhaps we’re persuaded to accept that in general we have:



Be careful to note that prime notation x´ = Λ-1x, here is defined opposite to what we used for translation, rotation whereby **r**´ = R**r**, or T**r**, not R-1**r** or T-1**r**. I’m not sure all of this is really accurate. I suspect it isn’t because there are ambiguities when trying to work out how a four vector like μ, or μ transform under this operator, because H technically commutes with t if we consider the position space representation of the operator, but not if we consider H to be equivalent to i∂/∂t. So this mixing of time/space is a little ad hoc. If we stick with the latter interpretation, then we have, out to first order,



and,



which *does* accord expectations to at least 2nd order, since:



So we seem to have, at least in some sense, that for instance:



Whatever, what we’ve done so far is usually just taken for granted so at least it’s better than nothing.

**Spinor Representation**

Now for the spinor representation of the Lorentz transformation. Again, instead of trying to work out a set of generators (4D matrices in spinor space) which satisfy those commutation relations, it’s easiest to try to work up from our knowledge of the special case of rotation. We may presume that:



and this analogously would tell us that:



which we may rewrite as:



and in terms of the Pauli spin matrices, this would look like,



Now we need to find the appropriate generalization. Reviewing the properties of the γμ operators/matrices, we may be tempted to hypothesize:



and this *is* the case. In any event one can verify that they do satisfy the requisite commutation relations. So we have:



where recall:



One can explicitly work this out in terms of the Pauli matrices. Employing some of their properties listed in the first file, we can show that the Weyl rep decouples into the following diagonal result:



where the subscript ‘L’ refers to ‘left’, not Lorentz, and the subscript ‘R’ refers to ‘right’. Specializing to just boosts, and employing those Pauli-matrix product identities, we can show that:



Similarly to how ΛS transformed spacetime vectors as tensors, we have the same phenomenon here. Namely, for instance,



where this ΛL does stand for the usual Lorentz transformation. Can check these are true sometime.

**Putting it all together**

So then, altogether we have:



**Lorentz Invariance**

Now let’s check for Lorentz invariance of the Dirac equation. In this framework, we’d want to know if the Lorentz boosted/rotated ket satisfies the Dirac equation if the unboosted/rotated one does. So start with presumption (x is 4-vector) that:



holds in any coordinate system. Then recalling from up above that <x|p(β)S(β)|ψ> = <Λ(-β)x|S(β)|ψ> = S(β)|ψ(Λ(-β)(x))>, do we have?



when we boost the ket to a (β,θ) frame? Let’s see:



So it checks out. It’s easier to see from the passive transformation point of view. Then let’s say we switch to a reference moving backwards at -β,-θ. Then do we have?



So we have it again. Or one more time. We start with:



Let’s change to reference frame moving backwards at rate β. The wavefunction spinor expressed in the new coordinate system will be:



And the derivative would be expressed as:



So filling these in:



Now we’ll multiply both sides by S(-β),



Now use the transformation property of γμ,



And now clearly these two matrices are inverses of each other, so:



So there we have it! The key thing to note is that unless γμ transforms as a tensor, the equation won’t be invariant, basically, because then the combination γμ∂μ won’t be a scalar.